J. DIFFERENTIAL GEOMETRY
63 (2003) 63-95

HARD LEFSCHETZ THEOREM FOR VALUATIONS, COMPLEX INTEGRAL GEOMETRY, AND UNITARILY INVARIANT VALUATIONS

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Abstract

We obtain new general results on the structure of the space of translation invariant continuous valuations on convex sets (a version of the hard Lefschetz theorem). Using these and our previous results we obtain explicit characterization of unitarily invariant translation invariant continuous valuations. It implies new integral geometric formulas for real submanifolds in Hermitian spaces generalizing the classical kinematic formulas in Euclidean spaces due to Poincaré, Chern, Santaló, and others.

0. Introduction

In this paper we obtain new results on the structure of the space of even translation invariant continuous valuations on convex sets. In particular we prove a version of hard Lefschetz theorem for them and introduce certain natural duality operator which establishes an isomorphism between the space of such valuations on a linear space V and on its dual V^* (with an appropriate twisting). Then we obtain an explicit geometric classification of unitarily invariant translation invariant continuous valuations on a Hermitian space $\mathbb{C}^n$. This classification is used to deduce new integral geometric formulas for real submanifolds in Hermitian spaces generalizing the classical kinematic formulas in Euclidean spaces due to Poincaré, Chern, Santaló, and others.

Let us describe the results in more details. First let us remind the definition of valuation. Let V be a finite dimensional real vector space. Let $\mathcal{K}(V)$ denote the class of all convex compact subsets of V.

Received 08/05/2002.

Definition.

a) A function $\phi : \mathcal{K}(V) \longrightarrow \mathbb{C}$ is called a valuation if for any $K_1,\ K_2 \in \mathcal{K}(V)$ such that their union is also convex one has
$$\phi(K_1 \cup K_2) = \phi(K_1) + \phi(K_2) - \phi(K_1 \cap K_2).$$

b) A valuation ϕ is called continuous if it is continuous with respect the Hausdorff metric on $\mathcal{K}(V)$.

Remind that the Hausdorff metric d_H on $\mathcal{K}(V)$ depends on the choice of a Euclidean metric on V and it is defined as follows: $d_H(A,B) := \inf\{\varepsilon > 0 | A \subset (B)_\varepsilon \text{ and } B \subset (A)_\varepsilon\}$, where $(U)_\varepsilon$ denotes the ε-neighborhood of a set U. Then $\mathcal{K}(V)$ becomes a locally compact space (by the Blaschke selection theorem).

In this paper we are interested only in translation invariant continuous valuations. The space of such valuations will be denoted by $\mathrm{Val}\,(V)$. The simplest examples of such valuations are a Lebesgue measure on V and the Euler characteristic χ (which is equal to 1 on each convex compact set). For the classical theory of valuations we refer to the surveys [39], [40]. For a brief overview of more recent results see [3] and [4].

Definition. A valuation ϕ is called homogeneous of degree k (or k-homogeneous) if for every convex compact set K and for every scalar $\lambda > 0$
$$\phi(\lambda K) = \lambda^k \phi(K).$$

Let us denote by $\mathrm{Val}_k(V)$ the space of translation invariant continuous k-homogeneous valuations.

Theorem (McMullen [38])**.**
$$\mathrm{Val}\,(V) = \bigoplus_{k=0}^{n} \mathrm{Val}_k(V),$$
where $n = \dim V$*.*

In particular note that the degree of homogeneity is an integer between 0 and $n = \dim V$. It is known that $\mathrm{Val}_0(V)$ is one-dimensional and it is spanned by the Euler characteristic χ, and $\mathrm{Val}_n(V)$ is also one-dimensional and is spanned by a Lebesgue measure [24]. The space $\mathrm{Val}_n(V)$ is also denoted by $|\wedge V^*|$ (or by $\mathrm{Dens}\,(V)$, the space of densities on V). Let us denote by $\mathrm{Val}^{\mathrm{ev}}(V)$ the subspace of $\mathrm{Val}\,(V)$ of even valuations (a valuation ϕ is called even if $\phi(-K) = \phi(K)$ for every $K \in \mathcal{K}(V)$). Similarly one defines the subspace $\mathrm{Val}^{\mathrm{odd}}(V)$ of odd valuations. One has further decomposition with respect to parity:

$$\mathrm{Val}_k(V) = \mathrm{Val}_k^{\mathrm{ev}}(V) \oplus \mathrm{Val}_k^{\mathrm{odd}}(V),$$

where $\mathrm{Val}_k^{\mathrm{ev}}(V)$ is the subspace of even k-homogeneous valuations, and $\mathrm{Val}_k^{\mathrm{odd}}(V)$ is the subspace of odd k-homogeneous valuations.

Let us fix on V a Euclidean metric, and let D denote the unit Euclidean ball with respect to this metric. Let us define on the space of translation invariant continuous valuations an operation Λ of mixing with the Euclidean ball D, namely

$$(\Lambda\phi)(K) := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \phi(K + \varepsilon D)$$

for any convex compact set K. Note that $\phi(K + \varepsilon D)$ is a polynomial in $\varepsilon \geq 0$ by McMullen's theorem [38]. It is easy to see that the operator Λ preserves parity and decreases the degree of homogeneity by one. In particular we have

$$\Lambda : \mathrm{Val}_k^{\mathrm{ev}}(V) \longrightarrow \mathrm{Val}_{k-1}^{\mathrm{ev}}(V).$$

To formulate our first main result we will need one more definition from the representation theory. Let G be a Lie group. Let ρ be a continuous representation of G in a Fréchet space F. A vector $v \in F$ is called *G-smooth* if the map $G \longrightarrow F$ defined by $g \longmapsto g(v)$ is infinitely differentiable. It is well-known (and easy to prove) that smooth vectors form a linear G-invariant subspace which is dense in F. We will denote it by F^{sm}. It is well-known (see e.g., [49]) that F^{sm} has a natural structure of a Fréchet space, and the representation of G in F^{sm} is continuous with respect to this topology. In our situation the Fréchet space $F = \mathrm{Val}(V)$ with the topology of uniform convergence on compact subsets of $\mathcal{K}(V)$, and $G = \mathrm{GL}(V)$. The action of $\mathrm{GL}(V)$ on $\mathrm{Val}(V)$ is the natural one, namely for any $g \in \mathrm{GL}(V)$, $\phi \in \mathrm{Val}(V)$ one has $(g(\phi))(K) = \phi(g^{-1}K)$.

The following result is a version of the hard Lefschetz theorem.

Theorem 1.1.1. *Let $n/2 < k \leq n$. Then*

$$\Lambda^{2k-n} : (\mathrm{Val}_k^{\mathrm{ev}}(V))^{\mathrm{sm}} \longrightarrow (\mathrm{Val}_{n-k}^{\mathrm{ev}}(V))^{\mathrm{sm}}$$

is an isomorphism. In particular for $1 \leq i \leq 2k - n$ the map

$$\Lambda^i : (\mathrm{Val}_k^{\mathrm{ev}}(V))^{\mathrm{sm}} \longrightarrow (\mathrm{Val}_{k-i}^{\mathrm{ev}}(V))^{\mathrm{sm}}$$

is injective.

Our terminology is motivated by the classical hard Lefschetz theorem (see e.g., [21]) about the cohomology of Kähler manifolds. To continue this analogy note that recently we have observed [5] the natural multiplicative structure on $(\mathrm{Val}\,(V))^{\mathrm{sm}}$ (see also [4]). More precisely this space has natural structure of commutative associative graded algebra (where the grading is given by the degree of homogeneity). It satisfies a version on the Poincaré duality with respect to these multiplication and grading.

The operator Λ turns out to be closely related to so called cosine transform on real Grassmannians, and the proof of Theorem 1.1.1 is based on the solution of the cosine transform problem by J. Bernstein and the author [6] (some particular cases of this problem were solved previously by Matheron [37] and Goodey, Howard, and Reeder [19]).

Our next main result establishes connection between even translation invariant continuous valuations on V and on its dual space V^*. In order to formulate it let us make an elementary remark from linear algebra. Let $E \subset V$ be any k-dimensional subspace. One has the canonical isomorphism $|\wedge^n V| = |\wedge^k E| \otimes |\wedge^{n-k} (V/E)|$. Note also that $V/E = (E^\perp)^*$. Hence we get the canonical isomorphism

$$|\wedge^k E^*| = |\wedge^{n-k} (E^\perp)^*| \otimes |\wedge^n V^*|.$$

Then we have:

Theorem 1.2.1. *For any $k = 0, 1, \ldots, n (= \dim V)$ there exists a natural isomorphism*

$$\mathbb{D} : (\mathrm{Val}_k^{\mathrm{ev}}(V))^{\mathrm{sm}} \xrightarrow{\sim} \left(\mathrm{Val}_{n-k}^{\mathrm{ev}}(V^*)\right)^{\mathrm{sm}} \otimes |\wedge^n V^*|.$$

This isomorphism $\mathbb{D}$ is uniquely characterized by the following property: let $\phi \in \mathrm{Val}_k^{\mathrm{ev}}(V)$ and let $E \in \mathrm{Gr}_k(V)$; then $\phi|_E = \mathbb{D}(\phi)|_{E^\perp}$ under the above identification $|\wedge^k E^*| = |\wedge^{n-k} (E^\perp)^*| \otimes |\wedge^n V^*|$.

The proof of this theorem uses the representation theoretical interpretation of the space $\mathrm{Val}^{\mathrm{ev}}(V)$ given in [2], where this space was characterized as the unique irreducible submodule of some standard $\mathrm{GL}(V)$-module with smallest Gelfand-Kirillov dimension (of the corresponding Harish-Chandra module).

Now let us discuss the translation invariant continuous valuations invariant under some group G of linear transformations of V. This space will be denoted by $\mathrm{Val}^G(V)$. If G is the group of orthogonal transformations $\mathrm{O}(n)$ or special orthogonal transformations $\mathrm{SO}(n)$ the corresponding space of valuations is described explicitly by the following famous result of H. Hadwiger.

Theorem (Hadwiger, [24]). *Let V be n-dimensional Euclidean space. The intrinsic volumes $V_0, V_1, \dots, V_n$ form a basis of* $\mathrm{Val}^{\mathrm{SO}(n)}(V)$ $(= \mathrm{Val}^{\mathrm{O}(n)}(V))$.

Let us remind the definition of the intrinsic volumes V_i. Let Ω be a compact (not necessarily convex) domain in a Euclidean space V with smooth boundary $\partial\Omega$. Let $n = \dim V$. For any point $s \in \partial\Omega$ let $k_1(s), \dots, k_{n-1}(s)$ denote the principal curvatures at s. For $0 \le i \le n-1$ define

$$V_i(\Omega) := \frac{1}{(n-i)\mathrm{vol}_{n-i}(D_{n-i})} \binom{n-1}{n-1-i}^{-1} \int_{\partial\Omega} \{k_{j_1}, \dots, k_{j_{n-1-i}}\} d\sigma,$$

where $\{k_{j_1}, \dots, k_{j_{n-1-i}}\}$ denotes the $(n-1-i)$-th elementary symmetric polynomial in the principal curvatures, $d\sigma$ is the measure induced on $\partial\Omega$ by the Euclidean metric, and D_{n-i} denotes the unit $(n-i)$-dimensional ball. It is well-known (see e.g., [44]) that V_i (uniquely) extends by continuity in the Hausdorff metric to $\mathcal{K}(V)$. Define also $V_n(\Omega) := \mathrm{vol}\,(\Omega)$. Note that V_0 is proportional to the Euler characteristic χ. It is well-known that $V_0, V_1, \dots, V_n$ belong to $\mathrm{Val}^{\mathrm{O}(n)}(V)$. It is easy to see that V_k is homogeneous of degree k.

Now let us describe unitarily invariant valuations on the Hermitian space $\mathbb{C}^n$. Let us denote by $\mathrm{IU}(n)$ the group of isometries of the Hermitian space $\mathbb{C}^n$ preserving the complex structure (thus $\mathrm{IU}(n) = \mathbb{C}^n \rtimes \mathrm{U}(n)$). Let ${}^{\mathbf{C}}\mathcal{A}\mathrm{Gr}_j$ denote the Grassmannian of affine complex subspaces of $\mathbb{C}^n$ of complex dimension j. Clearly ${}^{\mathbf{C}}\mathcal{A}\mathrm{Gr}_j$ is a homogeneous space of $\mathrm{IU}(n)$ and it has a unique (up to a constant) $\mathrm{IU}(n)$-invariant measure (called Haar measure). For every nonnegative integers p and k such that $2p \le k \le 2n$ let us introduce the following valuations:

$$U_{k,p}(K) = \int_{E \in {}^{\mathbf{C}}\mathcal{A}\mathrm{Gr}_{n-p}} V_{k-2p}(K \cap E) \cdot dE.$$

Then $U_{k,p} \in \mathrm{Val}_k^{\mathrm{U}(n)}(\mathbb{C}^n)$.

Theorem 2.1.1. *The valuations $U_{k,p}$ with $0 \le p \le \frac{\min\{k, 2n-k\}}{2}$ form a basis of the space* $\mathrm{Val}_k^{\mathrm{U}(n)}(\mathbb{C}^n)$.

This result is the Hermitian generalization of the (Euclidean) Hadwiger theorem. The proof of this theorem is highly indirect. It turns out to be necessary to study the $\mathrm{GL}_{2n}(\mathbb{R})$-module structure of the infinite dimensional space $\mathrm{Val}^{\mathrm{ev}}(\mathbb{C}^n)$. The proof of Theorem 2.1.1 uses most of the facts known about even valuations including the solution of the McMullen conjecture [2], cosine transform [6], the hard Lefschetz theorem for valuations, and the results of Howe and Lee [26] on the K-type structure of certain GL-modules.

Note that there are some other natural examples of valuations from $\mathrm{Val}^{\mathrm{U}(n)}(\mathbb{C}^n)$, for instance the averaged volume of projections of a convex set to all complex (or, say, Lagrangian) subspaces. Theorem 2.1.1 implies that all of them are linear combinations of $U_{k,p}$ with the above range of indices k, p. We would also like to mention another interesting example of such valuation which comes from the complex analysis. It is so called Kazarnovskii's pseudovolume. It was introduced and studied by B. Kazarnovskii [30], [31] in order to write down a formula for the number of zeros of a system of exponential sums in terms of their Newton polytopes. His results generalize in some sense the well-known results of D. Bernstein [7] and A. Kouchnirenko [35] on the number of zeros of a system of polynomial equations (see also [18]). We will recall the definition of Kazarnovskii's pseudovolume in Subsection 3.3. As a corollary of Theorem 2.1.1 we present a new formula for Kazarnovskii's pseudovolume in integral geometric terms (Theorem 3.3.2). It also seems that the valuation property of Kazarnovskii's pseudovolume was not mentioned previously in the literature.

The classification of unitarily invariant valuations is used to obtain new integral geometric formulas in the Hermitian space $\mathbb{C}^n$. Let us state some of them. Let Ω_1, Ω_2 be compact domains with smooth boundary in $\mathbb{C}^n$ such that $\Omega_1 \cap \mathrm{U}(\Omega_2)$ has finitely many components for all $U \in \mathrm{IU}(n)$. The new result is:

Theorem 3.1.1. *Let Ω_1, Ω_2 be compact domains in $\mathbb{C}^n$ with piecewise smooth boundaries such that for every $U \in \mathrm{IU}(n)$ the intersection $\Omega_1 \cap U(\Omega_2)$ has finitely many components. Then*

$$\int_{U\in \mathrm{IU}(n)} \chi(\Omega_1 \cap U(\Omega_2))dU = \sum_{k_1+k_2=2n} \sum_{p_1,p_2} \kappa(k_1,k_2,p_1,p_2) U_{k_1,p_1}(\Omega_1) U_{k_2,p_2}(\Omega_2),$$

where the inner sum runs over $0 \le p_i \le k_i/2$, $i = 1, 2$, and $\kappa(k_1, k_2, p_1, p_2)$ are certain uniquely defined constants depending on n, k_1, k_2, p_1, p_2 only.

The study of the left-hand side in this formulas was started by J. Fu [16].

Theorem 3.1.2. *Let Ω be a compact domain in $\mathbb{C}^n$ with piecewise smooth boundary. Let $0 < q < n$, $0 < 2p < k < 2q$. Then*

$$\int_{E\in {}^{\mathcal{C}}\mathcal{A}\mathrm{Gr}_q} U_{k,p}(\Omega\cap E) = \sum_{p=0}^{[k/2]+n-q} \gamma_p \cdot U_{k+2(n-q),p}(\Omega),$$

where the constants γ_p depend only on n, q, and p.

Let us denote by $\mathcal{A}\mathrm{LGr}(\mathbb{C}^n)$ the (noncompact) Grassmannian of *affine* Lagrangian subspaces of $\mathbb{C}^n$. Clearly it is a homogeneous space of the group $\mathrm{IU}(n)$ and hence has a Haar measure.

Theorem 3.1.3. *Let Ω be a compact domain in $\mathbb{C}^n$ with piecewise smooth boundary. Then*

$$\int_{\mathcal{A}\mathrm{LGr}(\mathbb{C}^n)} \chi(E\cap\Omega)dE = \sum_{p=0}^{[n/2]} \beta_p \cdot U_{n,p}(\Omega),$$

where β_p are certain uniquely defined constants depending on n and p only.

Theorems 3.1.1 and 3.1.2 are analogs of general kinematic formulas of Chern [11], [13] and Federer [15] (see also [43], especially Ch. 15, and [34]). Further generalizations in the Euclidean case were obtained by Cheeger, Müller, and Schrader [10] and J. Fu [16]. For more recent results in this direction and further references we refer to the recent survey by Hug and Schneider [27]. For classical results in Hermitian integral geometry we refer to [12], [20], [42]. In these papers the authors discuss the integral geometry of *complex* submanifolds. The integral geometry of Lagrangian submanifolds also was studied (see e.g., [36]). The integral geometry of real submanifolds in the complex projective space $\mathbb{C}P^n$ was studies by H. Tasaki [47], [48] and Kang and Tasaki [28], [29]. In these papers the authors obtain explicit Poincaré type formulas for real submanifolds of certain specific dimensions. Their results use in turn a general Poincaré type formula in Riemannian homogeneous spaces due to R. Howard [25].

It would be of interest to compute the constants $\kappa(k_1, k_2, p_1, p_2)$, γ_p, and β_p in Theorems 3.1.1, 3.1.2, and 3.1.3 explicitly. We could not do it in general. But we were able to compute them only in the first nontrivial case $n = 2$. One of such computations is presented in Subsection 3.2. Much more complete treatment of the integral geometric formulas (including computation of all constants) in $\mathbb{C}^2$, $\mathbb{C}^3$, and also in the 2- and 3-dimensional complex projective and hyperbolic spaces was done recently by H. Park in his thesis [41].

The paper is organized as follows. In Section 1 we discuss the results about the structure of the space of even translation invariant continuous valuations. Namely in Subsection 1.1 we prove the hard Lefschetz theorem for valuations and deduce some corollaries from it. In Subsection 1.2 we discuss the duality on valuations, in particular we prove Theorem 1.2.1. In Section 2 we prove the classification of unitarily invariant translation invariant continuous valuations. In Section 3 we discuss the integral geometry in complex spaces. In Subsection 3.1 we obtain the integral geometric formulas in $\mathbb{C}^n$. In Subsection 3.2 we compute explicitly the constants in one of such formulas in $\mathbb{C}^2$ (Theorem 3.2.4).

Acknowledgements. We express our gratitude to J. Bernstein for numerous useful discussions. We would also like to thank P. Biran, J. Fu, and L. Polterovich for useful conversations.

1. Hard Lefschetz theorem and duality for valuations

Let V be an n-dimensional real vector space. In Subsection 1.1 of this section we prove an analogue of the hard Lefschetz theorem for translation invariant even continuous valuations. In Subsection 1.2 we introduce the notion of a valuation dual to a given translation invariant even continuous valuation which satisfies some additional mild technical condition of $GL(V)$- smoothness (defined in the introduction). This construction uses the representation theoretical interpretation of the space of valuations given in [2]. The geometric examples will be given in Proposition 2.1.7 of Section 2.

1.1 An analogue of the hard Lefschetz theorem for valuations

The main result of this subsection is the following analogue of the hard Lefschetz theorem where the operator Λ was defined in the introduction.

Theorem 1.1.1. *Let $n \geq k > n/2$. Then $\Lambda^{2k-n} : (\mathrm{Val}_k^{ev}(V))^{sm} \longrightarrow (\mathrm{Val}_{n-k}^{ev}(V))^{sm}$ is an isomorphism. In particular $\Lambda^i : \mathrm{Val}_k^{ev}(V) \longrightarrow$*

$\mathrm{Val}_{k-i}^{\mathrm{ev}}(V)$ *is injective for* $1 \leq i \leq 2n-k$.

The proof of this theorem uses the cosine transform on real Grassmannians, thus we will remind first its definition and the relevant properties. We will denote by ${}^{\mathbf{R}}\mathrm{Gr}_j(V)$ the Grassmannian of real j-dimensional linear subspaces in V. Assume that $1 \leq i \leq j \leq n-1$. For two subspaces $E \in {}^{\mathbf{R}}\mathrm{Gr}_i(V)$, $F \in {}^{\mathbf{R}}\mathrm{Gr}_j(V)$ let us define the *cosine of the angle* between E and F:

$$|\cos(E,F)| := \frac{\mathrm{vol}_i(\mathrm{Pr}_F(A))}{\mathrm{vol}_i(A)},$$

where A is any subset of E of nonzero volume, Pr_F denotes the orthogonal projection onto F, and vol_i is the i-dimensional measure induced by the Euclidean metric. (Note that this definition does not depend on the choice of a subset $A \subset E$). In the case $i \geq j$ we define the cosine of the angle between E and F as cosine of the angle between their orthogonal complements:

$$|\cos(E,F)| := |\cos(E^{\perp}, F^{\perp})|.$$

(It is easy to see that for $i = j$ both definitions coincide.)

For any $1 \leq i,\, j \leq n-1$ one defines the cosine transform

$$T_{j,i} : C({}^{\mathbf{R}}\mathrm{Gr}_i(V)) \longrightarrow C({}^{\mathbf{R}}\mathrm{Gr}_j(V))$$

as follows:

$$(T_{j,i}f)(F) := \int_{{}^{\mathbf{R}}\mathrm{Gr}_i(V)} |\cos(E,F)| f(E) dE,$$

where the integration is with respect to the Haar measure on the Grassmannian such that the total measure is equal to 1. Clearly the cosine transform commutes with the action of the orthogonal group $\mathrm{O}(n)$, and hence its image is an $\mathrm{O}(n)$-invariant subspace of functions.

Now let us recall the imbedding $\mathrm{Val}_k^{\mathrm{ev}}(V) \longrightarrow C({}^{\mathbf{R}}\mathrm{Gr}_k(V))$ which we will call the Klain imbedding. Let $\phi \in \mathrm{Val}_k^{\mathrm{ev}}(V)$. For every $E \in {}^{\mathbf{R}}\mathrm{Gr}_k(V)$ let us consider the restriction of ϕ to all convex compact subsets of E. This is an even translation invariant valuation homogeneous of degree k. Hence, by a result due to Hadwiger [24], it is a density on E (i.e., a Lebesgue measure). Thus it is equal to $f(E) \cdot \mathrm{vol}_E$, where vol_E is the volume form on E defined by the metric on V, and $f(E)$ is a constant depending on E. Thus $\phi \mapsto f$ defines the map $\mathrm{Val}_k^{\mathrm{ev}}(V) \longrightarrow C({}^{\mathbf{R}}\mathrm{Gr}_k(V))$ which turns out to be an imbedding by a result due to D. Klain ([32]; this result was stated in this form in [33] and in [1]). Let us denote this image by I_k. Moreover it was shown in [6] that the image of the Klain imbedding coincides with the image of the cosine transform $T_{k,k} : C({}^{\mathbf{R}}\mathrm{Gr}_k(V)) \longrightarrow C({}^{\mathbf{R}}\mathrm{Gr}_k(V))$ (at least on the level of $\mathrm{GL}(V)$-smooth vectors).

Lemma 1.1.2. *Let $k \geq n/2$. The cosine transform*

$$T_{n-k,k} : C({}^{\boldsymbol{R}}\mathrm{Gr}_k(V)) \longrightarrow C({}^{\boldsymbol{R}}\mathrm{Gr}_{n-k}(V))$$

maps I_k to I_{n-k} and induces isomorphism of $\mathrm{O}(n)$-smooth vectors of these subspaces.

Proof. It is well-known that for admissible $\mathrm{GL}(V)$-modules of finite length the subspaces of $\mathrm{GL}(V)$-smooth and $\mathrm{O}(n)$-smooth vectors coincide (more generally, $\mathrm{GL}(V)$ can be replaced by any real reductive group G, and $\mathrm{O}(n)$ can be replaced by a maximal compact subgroup of G). First let us prove that I_k and I_{n-k} have the same decomposition under the action of the orthogonal group $\mathrm{O}(n)$. Indeed the correspondence $E \mapsto E^{\perp}$ induces an isomorphism $S : C({}^{\mathbf{R}}\mathrm{Gr}_k(V)) \longrightarrow C({}^{\mathbf{R}}\mathrm{Gr}_{n-k}(V))$ commuting with the action of $\mathrm{O}(n)$. Moreover we have the following relation between the cosine transforms:

$$T_{n-k,n-k} = S T_{k,k} S^{-1}.$$

Hence it follows that $S((I_k)^{\mathrm{sm}}) = (I_{n-k})^{\mathrm{sm}}$ (it is immediate on the level of $\mathrm{O}(n)$-finite vectors; to deduce it for $\mathrm{O}(n)$-smooth vectors one should use the Casselman-Wallach theorem [9] as it is done in [6]).

Next it is well-known (see e.g., [6], Lemma 1.7) that the cosine transform $T_{n-k,k}$ can be written (up to a nonzero normalizing constant which we ignore) as a composition $T_{n-k,n-k} \circ R_{n-k,k}$, where $R_{n-k,k} : C({}^{\mathbf{R}}\mathrm{Gr}_k(V)) \longrightarrow C({}^{\mathbf{R}}\mathrm{Gr}_{n-k}(V))$ is the Radon transform. It was shown in [17] that

$$R_{n-k,k} : C^{\infty}({}^{\mathbf{R}}\mathrm{Gr}_k(V)) \longrightarrow C^{\infty}({}^{\mathbf{R}}\mathrm{Gr}_{n-k}(V))$$

is an isomorphism. We claim that $R_{n-k,k}((I_k)^{\mathrm{sm}}) = (I_{n-k})^{\mathrm{sm}}$. To see this remind that the quasiregular representation of $\mathrm{O}(n)$ in the space of functions on the Grassmannians is multiplicity free (since the Grassmannians are symmetric spaces). Hence it follows that two $\mathrm{O}(n)$-invariant closed subspaces of $C^{\infty}({}^{\mathbf{R}}\mathrm{Gr}_{n-k}(V))$ have the same $\mathrm{O}(n)$-finite vectors if and only if these subspaces have the same decomposition under the action of $\mathrm{O}(n)$ (in the abstract sense). Hence $R_{n-k,k}(I_k)$ and I_{n-k} have the same $\mathrm{O}(n)$-finite vectors. The coincidence of $\mathrm{O}(n)$-smooth vectors follows again from the Casselman-Wallach theorem [9] and the fact that the Radon transform can be rewritten as an intertwining operator of admissible $\mathrm{GL}(V)$-modules of finite length (see [17]).

Since $T_{n-k,n-k}$ is selfadjoint its restriction to I_{n-k} has trivial kernel and dense image. But the key observation of [6] was that $T_{n-k,n-k}$ can be rewritten as an intertwining operator of certain $\mathrm{GL}(V)$-modules. This and the Casselman-Wallach theorem [9] imply that

$$T_{n-k,n-k}((I_{n-k})^{\mathrm{sm}}) = (I_{n-k})^{\mathrm{sm}}.$$

Hence $T_{n-k,k}((I_k)^{\mathrm{sm}}) = (I_{n-k})^{\mathrm{sm}}$. q.e.d.

Now let us prove Theorem 1.1.1.

Proof of Theorem 1.1.1. Since the image of $\mathrm{GL}(V)$-smooth continuous k-homogeneous valuations in $C({}^{\mathbf{R}}\mathrm{Gr}_k(V))$ coincides with the image of the cosine transform on $\mathrm{GL}(V)$-smooth functions, then every $\mathrm{GL}(V)$-smooth valuation $\phi \in \mathrm{Val}_k^{\mathrm{ev}}(V)$ can be represented in the form

$$\phi(K) = \int_{{}^{\mathbf{R}}\mathrm{Gr}_k(V)} f(E)\mathrm{vol}_k(\mathrm{Pr}_E(K))dE,$$

where f is a smooth function on ${}^{\mathbf{R}}\mathrm{Gr}_k(V)$, K is an arbitrary convex compact set, Pr_E denotes the orthogonal projection onto E, and the integration is with respect to the Haar measure on the Grassmannian. Moreover for every smooth function f, the expression defined by this formula is a valuation from $(\mathrm{Val}_k^{\mathrm{ev}}(V))^{\mathrm{sm}}$. For a given valuation ϕ the function f is not defined uniquely. But we can choose $f \in I_k$, i.e., in the image of the cosine transform; then it will be defined uniquely. So we will assume that $f \in I_k$. Let us apply Λ^{2k-n} to it. Then it is easy to see that

$$(\Lambda^{2k-n}\phi)(K) = c \cdot \int_{{}^{\mathbf{R}}\mathrm{Gr}_k(V)} f(E)V_{n-k}(\mathrm{Pr}_E(K))dE,$$

where c is a nonzero normalizing constant, and $V_{n-k}(\mathrm{Pr}_E(K))$ denotes the $(n-k)$-th intrinsic volume of $\mathrm{Pr}_E(K)$ inside E, i.e., it is the mixed volume of $\mathrm{Pr}_E(K)$ taken $n-k$ times with the unit ball of E taken $2k-n$ times. The image g of $\Lambda^{2k-n}\phi$ in functions on the Grassmannian $C({}^{\mathbf{R}}\mathrm{Gr}_{n-k}(V))$ can be described as follows. It is easy to see that for every subspace $F \in {}^{\mathbf{R}}\mathrm{Gr}_{n-k}(V)$

$$g(F) = c' \cdot \int_{{}^{\mathbf{R}}\mathrm{Gr}_{n-k}(V)} f(E)|\cos(F,E)|dE,$$

where c' is a nonzero normalizing constant. Namely g is equal (up to a normalization) to the cosine transform $T_{n-k,k}(f)$ of f. By Lemma 1.1.2 $T_{n-k,k}$ induces the isomorphism between $\mathrm{GL}(V)$-smooth vectors of I_k and of I_{n-k}. This proves Theorem 1.1.1. q.e.d.

For a subgroup $G \subset \mathrm{GL}(V)$ let us denote by $\mathrm{Val}_k^G(V)$ the space of translation invariant G-invariant k-homogeneous continuous valuations. Let $h_k := \dim \mathrm{Val}_k^G(V)$.

Corollary 1.1.3. *Let G be a compact subgroup of the orthogonal group which acts transitively on the unit sphere and contains the operator $-Id$. Then $\mathrm{Val}_k^G(V)$ is a finite dimensional space, and for $n/2 < k \leq n$*

$$\Lambda^{2k-n} : \mathrm{Val}_k^G(V) \longrightarrow \mathrm{Val}_{n-k}^G(V)$$

is an isomorphism. Consequently the numbers h_i satisfy the Lefschetz inequalities:

$$h_i \leq h_{i+1} \text{ for } i < n/2, \text{ and } h_i = h_{n-i} \text{ for } i = 0, \dots, n.$$

Proof. The finite dimensionality of $\mathrm{Val}_k^G(V)$ was proved in [1]. Let us show that this implies that all vectors from $\mathrm{Val}_k^G(V)$ are $\mathrm{O}(n)$-finite (in particular $\mathrm{GL}(V)$-smooth). Indeed let Z be the minimal closed $\mathrm{O}(n)$-invariant subspace of the space $\mathrm{Val}_k(V)$ containing $\mathrm{Val}_k^G(V)$. The space Z is decomposed under the action of $\mathrm{O}(n)$ into the direct sum of irreducible components, and each component enters with finite multiplicity (since the space of translation invariant continuous valuations of the given degree of homogeneity and parity can be realized as a subquotient of a representation of $\mathrm{GL}(V)$ induced from a character of a parabolic subgroup, see Section 2 in [2]). Thus let $Z = \oplus_i \rho_i$ be this decomposition. We have a continuous projection $\pi : \mathrm{Val}_k(V) \longrightarrow \mathrm{Val}_k^G(V)$ defined by $\pi(\phi) = \int_{g \in G} g(\phi) dg$. Clearly $\mathrm{Im}(\pi) = \mathrm{Val}_k^G(V) = (Z)^G$. But $(Z)^G = \oplus_i (\rho_i)^G$. Since $\mathrm{Val}_k^G(V)$ is finite dimensional, $(\rho_i)^G = 0$ for all but finitely many i's. In other words there is a finite set of indices A such that $\mathrm{Val}_k^G(V) \subset \oplus_{i \in A} \rho_i$. Thus all elements of $\mathrm{Val}_k^G(V)$ are $\mathrm{O}(n)$-finite.

Next obviously $\Lambda(\mathrm{Val}_k^G(V)) \subset \mathrm{Val}_{k-1}^G(V)$. The rest follows from Theorem 1.1.1. q.e.d.

1.2 Duality on valuations

Let V be an n-dimensional real vector space. Let us denote by V^* its dual space. Let us denote by $|\wedge^n V^*|$ the (one-dimensional) space of complex-valued Lebesgue measures on V. Let us consider the space $\mathrm{Val}_k^{\mathrm{ev}}(V^*)\otimes|\wedge^n V^*|$ of translation invariant even continuous k-homogeneous valuations on V^* with values in $|\wedge^n V^*|$. Note that on both spaces we have the natural (continuous) representation of the group $\mathrm{GL}(V)$.

Before we state the main result of this subsection let us make a remark. For any subspace $E\in \mathrm{Gr}_k(V)$ consider the short exact sequence $0\longrightarrow E\longrightarrow V\longrightarrow V/E\longrightarrow 0$. From this sequence one gets the canonical isomorphism $|\wedge^n V|=|\wedge^k E|\otimes|\wedge^{n-k}(V/E)|$. Note also that $V/E=(E^\perp)^*$. Hence we get the canonical isomorphism

$$|\wedge^k E^*|=|\wedge^{n-k}(E^\perp)^*|\otimes|\wedge^n V^*|.$$

The main result of this subsection is:

Theorem 1.2.1. *For any $k=0,1,\dots,n$ there exists a natural isomorphism*

$$\mathbb{D}:(\mathrm{Val}_k^{\mathrm{ev}}(V))^{\mathrm{sm}}\xrightarrow{\sim}\left(\mathrm{Val}_{n-k}^{\mathrm{ev}}(V^*)\right)^{\mathrm{sm}}\otimes|\wedge^n V^*|.$$

This isomorphism $\mathbb{D}$ is defined uniquely by the following property: let $\phi\in(\mathrm{Val}_k^{\mathrm{ev}}(V))^{\mathrm{sm}}$ and let $E\in\mathrm{Gr}_k(V)$; then $\phi|_E=\mathbb{D}(\phi)|_{E^\perp}$ under the above identification $|\wedge^k E^*|=|\wedge^{n-k}(E^\perp)^*|\otimes|\wedge^n V^*|$.

Proof. First let us rewrite the Klain imbedding we have discussed in Subsection 1.1 of the space of even valuations $\mathrm{Val}_k^{\mathrm{ev}}(V)$ to functions on the Grassmannian in the notation which does not use any Euclidean structure. Instead of functions on the Grassmannian we have to consider sections of certain line bundle L_k over the Grassmannian ${}^{\mathbf{R}}\mathrm{Gr}_k(V)$. The fiber of L_k over a subspace $E\in{}^{\mathbf{R}}\mathrm{Gr}_k(V)$ is the (one-dimensional) space $|\wedge^k E^*|$ of complex valued Lebesgue measures on E. Clearly L_k is naturally $\mathrm{GL}(V)$-equivariant. Let us denote by $C({}^{\mathbf{R}}\mathrm{Gr}_k,L_k)$ the space of continuous sections of L_k. The map we have described in Subsection 1.1 can be rewritten as follows. Fix a valuation $\phi\in\mathrm{Val}_k^{\mathrm{ev}}(V)$. For any $E\in{}^{\mathbf{R}}\mathrm{Gr}_k(V)$ let us consider the restriction of ϕ to E. As previously, since this restriction $\phi|_E$ has maximal degree of homogeneity (equal to k) by Hadwiger's theorem [24] $\phi|_E$ is a Lebesgue measure on E. Thus ϕ defines a continuous section of L_k. As we have mentioned, the constructed map is injective. One of the main results of [2] says that the image of $\mathrm{Val}_k^{\mathrm{ev}}(V)$ in $C({}^{\mathbf{R}}\mathrm{Gr}_k(V),L_k)$ under this map is the unique "small" irreducible $\mathrm{GL}(V)$-submodule (Theorem 1.3 combined with Theorem 3.1 in [2]) . Let us give some comments what does it mean "small". First replace all $\mathrm{GL}(V)$-modules by their Harish-Chandra modules which are purely algebraic objects. For each Harish-Chandra module one defines an *associated variety* (or Bernstein's variety) which is an algebraic subvariety of the Lie algebra $gl_n(\mathbb{C})$, where $n = \dim(V)$. For details we refer to [8]. When we say that a given irreducible submodule A of a module B is "small" it means that the dimensions of the associated varieties of all other irreducible subquotients of B are strictly greater than that of A. Note also that the dimension of the associated variety of A is equal to the Gelfand-Kirillov dimension of the underlying Harish-Chandra module.

Now let us continue constructing the isomorphism $\mathbb{D}$. Let us consider the line bundle M_k over ${}^{\mathbf{R}}\mathrm{Gr}_{n-k}(V^*)$ the fiber of which over any $F \in {}^{\mathbf{R}}\mathrm{Gr}_{n-k}(V^*)$ is equal to $|\wedge^{n-k} F^*| \otimes |\wedge^n V^*|$ (note that $|\wedge^{n-k} F^*|$ is identified with the space of Lebesgue measures on F). As previously, $\mathrm{Val}_{n-k}^{\mathrm{ev}}(V^*) \otimes |\wedge^n V^*|$ can be realized as the only "small" irreducible submodule of $C({}^{\mathbf{R}}\mathrm{Gr}_{n-k}(V^*), M_k)$ (indeed these spaces differ from the previous two only by the twist by $|\wedge^n V^*|$). Hence it is sufficient to present the natural isomorphism between $C^\infty({}^{\mathbf{R}}\mathrm{Gr}_k(V), L_k)$ and $C^\infty({}^{\mathbf{R}}\mathrm{Gr}_{n-k}(V^*), M_k)$ where C^∞ denotes the space of C^∞-sections of the bundles. Let us do it. Let $E \in {}^{\mathbf{R}}\mathrm{Gr}_k(V)$. As previously, we have the canonical isomorphism

$$|\wedge^k E^*| = |\wedge^{n-k} (E^\perp)^*| \otimes |\wedge^n V^*|.$$

The correspondence $E \longmapsto E^\perp$ and the last identification give the desired isomorphism. q.e.d.

Now let us assume that V is a Euclidean space, i.e., on V we are given a positive definite quadratic form. This gives us the identification of V with its dual space V^*, and the identification of the space $|\wedge^n V^*|$ of Lebesgue measures on V with the complex line $\mathbb{C}$ (such that $1 \in \mathbb{C}$ corresponds to the Lebesgue measure on V which is equal to 1 on the unit cube). Also for any subspace E let us denote by vol_E the Lebesgue measure on E which is equal to 1 on the unit cube. Under these identifications we get

$$\mathbb{D} : (\mathrm{Val}_k^{\mathrm{ev}}(V))^{\mathrm{sm}} \tilde{\longrightarrow} (\mathrm{Val}_{n-k}^{\mathrm{ev}}(V))^{\mathrm{sm}}.$$

For this operator we have the following result which can be easily deduced from the last theorem.

Theorem 1.2.2. *Let V be an n-dimensional Euclidean space. Then for any $k = 0, 1, \dots, n$*

$$\mathbb{D} : (\mathrm{Val}_k^{\mathrm{ev}}(V))^{\mathrm{sm}} \tilde{\longrightarrow} (\mathrm{Val}_{n-k}^{\mathrm{ev}}(V))^{\mathrm{sm}}$$

is an isomorphism and $\mathbb{D}^2 = Id$. This operator $\mathbb{D}$ is defined uniquely by the following property: let $\phi \in \mathrm{Val}_k^{\mathrm{ev}}(V)$ and let $E \in \mathrm{Gr}_k(V)$; if $\phi|_E = f(E) \cdot \mathrm{vol}_E$ then $\mathbb{D}\phi|_{E^\perp} = f(E) vol_{E^\perp}$. Also $\mathbb{D}$ commutes with the action of $\mathrm{O}(n)$.

Example. Let χ denote the Euler characteristic on a Euclidean space V. Clearly $\chi \in \mathrm{Val}_0(V)$. Then $\mathbb{D}(\chi) = \mathrm{vol}_V$, and $\mathbb{D}(\mathrm{vol}_V) = \chi$.

2. Unitarily invariant valuations

In this section we will describe unitarily invariant translation invariant continuous valuations on convex compact subsets of $\mathbb{C}^n$ by writing down explicitly a basis in this space. Let k, l be integers such that $0 \le k \le 2n$ and $k/2 \le l \le n$. Let us define a valuation

$$C_{k,l}(K) := \int_{{}^{\mathbf{C}}\mathrm{Gr}_{l,n}} V_k(\mathrm{Pr}_F(K)) dF,$$

where the integration is with respect to the Haar measure on the complex Grassmannian of complex l-dimensional subspaces in $\mathbb{C}^n$, Pr_F denotes the orthogonal projection onto F, and $V_k(\mathrm{Pr}_F(K))$ denotes the k-th intrinsic volume of $\mathrm{Pr}_F(K)$ inside F, namely it is the mixed volume of $\mathrm{Pr}_F(K)$ taken k times with the unit Euclidean ball in F taken $2l - k$ times. Clearly $C_{k,l} \in \mathrm{Val}_k^{\mathrm{U}(n)}(\mathbb{C}^n)$. Note that for $l = n$ we get the usual intrinsic volumes. For $k = 0$ we get the Euler characteristic, and for $k = 2n$, $l = n$ we get the Lebesgue measure. Our next main result is:

Theorem 2.1.1. *Let k be an integer, $0 \le k \le 2n$. The dimension of the space $\mathrm{Val}_k^{\mathrm{U}(n)}(\mathbb{C}^n)$ is equal to $1 + \min\{[k/2], [(2n-k)/2]\}$. The valuations $C_{k,l}$ with $\frac{\max\{k, 2n-k\}}{2} \le l \le n$, form a basis of $\mathrm{Val}_k^{\mathrm{U}(n)}(\mathbb{C}^n)$.*

Remark. Later on in this section we will present another basis in the space of unitarily invariant valuations. This basis will be more convenient for the applications in integral geometry and for non-convex sets. In fact the connection between these two bases is not quite trivial and leads to new integral geometric formulas. This material will be discussed in more detail in Section 3.

Proof. The dimension of $\mathrm{Val}_k^{\mathrm{U}(n)}(\mathbb{C}^n)$ was computed in [2]. Hence it remains to show that the valuations $C_{k,l}$ with $\frac{\max\{k,2n-k\}}{2} \leq l \leq n$ are linearly independent. First of all it is clear that $C_{k,l} = c \cdot \Lambda(C_{k+1,l})$, where Λ is the operator from the hard Lefschetz theorem (Theorem 1.1.1), and c is a nonzero constant depending on n, k, l only. Hence by the hard Lefschetz theorem for unitarily invariant valuations (Corollary 1.1.3) the statement is reduced to the case $k \geq n$. Let us prove this case. We will prove the statement by induction in $2n-k$. If $2n-k=0$ then the result is clear since by [24] any translation invariant continuous N-homogeneous valuation on $\mathbb{R}^N$ is a Lebesgue measure. Now assume that $n \leq k < 2n$, and the theorem is true for valuations homogeneous of degree $> k$. If k is odd then the induction assumption, Corollary 1.1.3, and the computation of the dimension of unitarily invariant k-homogeneous valuations imply the result. Hence let us assume that k is even. Again using Corollary 1.1.3 it is sufficient to check that $C_{k,k/2}$ can not be presented as a linear combination of valuations $C_{k,l}$ with $l > \frac{k}{2}$.

In order to prove it, we will show that the special orthogonal group $\mathrm{SO}(2n)$ acts differently on $C_{k,k/2}$ and on $C_{k,l}$ with $l > \frac{k}{2}$. To formulate this more precisely let us introduce some notation. First recall that the set of highest weights of $\mathrm{SO}(2n)$ is parameterized by sequences of integers $\mu_1, \dots, \mu_{n-1}, \mu_n$ such that $\mu_1 \geq \dots \geq \mu_{n-1} \geq |\mu_n|$. For $1 \leq l \leq n$ let us denote by $\Lambda(l)$ the subset of highest weights of $\mathrm{SO}(2n)$ such that all μ_i's are even and if $l < n$ satisfy in addition the following condition: $\mu_j = 0$ for $j > l$.

The following result was proved in [2], Proposition 6.3.

Lemma 2.1.2. *The natural representation of* $\mathrm{SO}(2n)$ *in the space* $\mathrm{Val}_k^{\mathrm{ev}}(\mathbb{C}^n)$ *is multiplicity free and is isomorphic to a direct sum of irreducible components with highest weights* $(\mu_1, \mu_2, \dots, \mu_n) \in \Lambda(\min(k, 2n-k))$ *such that* $|\mu_2| \leq 2$.

Note that the explicit description of the K-type structure was heavily based on the results of Howe and Lee [26].

The next result is well-known (see e.g., [46], §8).

Lemma 2.1.3. *In every irreducible representation of* $\mathrm{SO}(2n)$ *the subspace of* $\mathrm{U}(n)$*-invariant vectors is at most* 1*-dimensional. This subspace is* 1*-dimensional if and only if the highest weight of the irreducible representation of* $\mathrm{SO}(2n)$ *is of the form* $(\mu_1, \dots, \mu_n)$ *where:*

(i) *if n is even then*

$$\mu_1 = \mu_2 \geq \mu_3 = \mu_4 \geq \cdots \geq \mu_{n-1} = \mu_n \geq 0;$$

(ii) *if n is odd then*

$$\mu_1 = \mu_2 \geq \mu_3 = \mu_4 \geq \cdots \geq \mu_{n-2} = \mu_{n-1} \geq \mu_n = 0.$$

The following lemma and Corollary 1.1.3 obviously imply Theorem 2.1.1.

Lemma 2.1.4. *Let k be even, $n \leq k < 2n$.*

(i) *The valuations $C_{k,l}$ with $l > k/2$ belong to the sum of the representations with highest weights $\mu \in \Lambda(2n-k-1)$.*

(ii) *The valuation $C_{k,k/2}$ does not belong to the above sum.*

Proof. First let us prove Part (i) of the lemma. As we have mentioned earlier $C_{k,l} = c \cdot \Lambda(C_{k+1,l})$ if $l > k/2$. Since the operator Λ commutes with the action of $\mathrm{SO}(2n)$ on valuations then it is sufficient to check that $C_{k+1,l}$ belongs to the sum of irreducible components with highest weights from $\Lambda(2n-k-1)$. As it was mentioned in Section 1 of this paper the space $\mathrm{Val}_{k+1}^{ev}(\mathbb{C}^n)$ can be imbedded into the space of continuous functions $C({}^{\mathbf{R}}\mathrm{Gr}_{k+1,2n})$. But it is well-known that all irreducible representations of $\mathrm{SO}(2n)$ which appear in the last space belong to $\Lambda(2n-k-1)$ (see e.g., [46] §8 for the general case of compact symmetric spaces). This proves Part (i) of the lemma.

Let us prove Part (ii) which is somewhat more computational. We will show that the image of $C_{k,k/2}$ in $C({}^{\mathbf{R}}\mathrm{Gr}_{k,2n})$ is not orthogonal to the irreducible subspace in $C({}^{\mathbf{R}}\mathrm{Gr}_{k,2n})$ with highest weight $(\underbrace{2,2,\dots,2}_{2n-k \text{ times}}, 0,\dots,0)$. Clearly this will finish the proof of Lemma 2.1.4, and hence the proof of Theorem 2.1.1.

From the definition of $C_{k,k/2}$ we immediately see that its image in $C({}^{\mathbf{R}}\mathrm{Gr}_{k,2n})$ is the function f such that

$$f(E) = c \cdot \int_{{}^{\mathbf{C}}\mathrm{Gr}_{k/2,n}} |\cos(E,F)| dF,$$

where c is a nonzero normalizing constant. In other words f is proportional to the cosine transform of the δ-function of the submanifold ${}^{\mathbf{C}}\mathrm{Gr}_{k/2,n} \subset {}^{\mathbf{R}}\mathrm{Gr}_{k,2n}$. We will denote it by $\delta_{{}^{\mathbf{C}}\mathrm{Gr}}$.

Lemma 2.1.5. *Let $k > n$ be even. Then $\delta_{\mathbf{c}_{\mathrm{Gr}}}$ is not orthogonal to the irreducible subspace in $C({}^{\mathbf{R}}\mathrm{Gr}_{k,2n})$ with highest weight*

$$(\underbrace{2, 2, \dots, 2}_{2n-k \text{ times}}, 0, \dots, 0).$$

First let us deduce our statement from this lemma. The cosine transform commutes with the action of $\mathrm{SO}(2n)$ on $C({}^{\mathbf{R}}\mathrm{Gr}_{k,2n})$. Hence by the Schur lemma it acts on each irreducible subspace as a multiplication by a scalar. Hence an irreducible subspace is contained in the image of the cosine transform if and only if the cosine transform on it does not vanish. However by Lemma 2.1.2 the irreducible subspace with the highest weight vector $(\underbrace{2, 2, \dots, 2}_{2n-k \text{ times}}, 0, \dots, 0)$ is contained in the image of $\mathrm{Val}^{\mathrm{ev}}_{2n,k}$ in $C({}^{\mathbf{R}}\mathrm{Gr}_{k,2n})$, and this image coincides with the image of the cosine transform by Theorem 1.1.3 of [6]. Thus it remains to prove Lemma 2.1.5 to finish the proof of Theorem 2.1.1.

Proof of Lemma 2.1.5. First observe that the statement of the lemma is purely representation theoretical. So replacing each subspace by its orthogonal complement we may and will assume that $k \leq n$ (oppositely to our previous assumption on k). Under this assumption it is easier to write down explicit formulas. It is sufficient to prove that $\delta_{\mathbf{c}_{\mathrm{Gr}}}$ is not orthogonal to the highest weight vector in the relevant irreducible subspace. This statement will be proven by a computation involving explicit form of the highest weight vector. First we will write it down following [45] (see also [22]).

Let $e_{i,j}$ denote $(2n \times 2n)$-matrix which has zeros at all but one place (i, j) where it has 1. Let us fix a Cartan subalgebra of $so(2n)$ spanned by $\{C_i\}_{i=1}^n$ where

$$C_i = e_{2i-1,2i} - e_{2i,2i-1},\ i = 1, \dots, n.$$

For any subspace $E \in {}^{\mathbf{R}}\mathrm{Gr}_{k,2n}$ let us choose an orthonormal basis $X^1, \dots, X^k$, and let us write its coordinates in the standard basis in columns of $2n \times k$-matrix:

$$\left| \begin{array}{ccc} X^1_1 & \dots & X^k_1 \\ \multicolumn{3}{c}{\dots\dots\dots\dots\dots} \\ X^1_{2n} & \dots & X^k_{2n} \end{array} \right|.$$

Let X_j denote the j-th row of this matrix. For $l \leq n$ let $A(l)$ be $l \times k$-matrix whose j-th row is $X_{2j-1} + \sqrt{-1} X_{2j}$, $j = 1, \dots, l$. The next lemma was proved in [45], Theorem 5 (see also [22]).

Lemma 2.1.6. *Let $k \leq n$. Let $(2m_1, 2m_2, \dots, 2m_k, 0, \dots, 0)$ be the highest weight of* $\mathrm{SO}(2n)$ *with $m_1 \geq m_2 \geq \dots \geq m_k \geq 0$. The irreducible subspace of $C({}^{\mathbf{R}}\mathrm{Gr}_{k,2n})$ with this highest weight has the highest weight vector of the form*

$$f_{m_1,\dots,m_k} = \det[A(1)\cdot A(1)^t]^{m_1-m_2}\cdot \det[A(2)\cdot A(2)^t]^{m_2-m_3}\cdot \\ \dots \cdot \det[A(k)\cdot A(k)^t]^{m_k}.$$

Recall that we are interested in the highest weight $(\underbrace{2,2,\dots,2}_{k \text{ times}}, 0,\dots,0)$. Hence the highest weight vector $F \in C({}^{\mathbf{R}}\mathrm{Gr}_{k,2n})$ has the form:

$$F := \det[A(k)\cdot A(k)^t].$$

Let us denote for brevity $m := k/2$ (recall that m is an integer). We have to show that

$$\int_{M\in {}^{\mathbf{C}}\mathrm{Gr}_{m,n}} F(M)dM \neq 0.$$

In fact we will show that the function F is nonnegative on ${}^{\mathbf{C}}\mathrm{Gr}_{m,n}$ and is not identically zero.

Let us choose in our hermitian space $\mathbb{C}^n$ an orthonormal hermitian basis $e_1, \dots, e_n$. Then in the realization $\mathbb{R}^{2n}$ of this space we will choose the basis

(∗)

$$e_1, e_2, e_3, \dots, e_k; \sqrt{-1}e_1, -\sqrt{-1}e_2, \sqrt{-1}e_3, \dots, -\sqrt{-1}e_k, \text{ other vectors.}$$

Fix any $E \in {}^{\mathbf{C}}\mathrm{Gr}_{m,n}$. Let us choose in E an orthonormal hermitian basis $\xi_1, \dots, \xi_m$. Then

$$\xi_t = \sum_{j=1}^n z_t^j e_j = \sum_{j=1}^n (\mathrm{Re} z_t^j \cdot e_j + \mathrm{Im} z_t^j \cdot (\sqrt{-1}e_j)),$$

with $z_t^j \in \mathbb{C}$. Then the vectors $\xi_1, \dots, \xi_m, \sqrt{-1}\xi_1, \dots, \sqrt{-1}\xi_m$ form a real basis of E. Let us write the coordinates of these vectors with respect to the basis $(*)$ in columns of the following matrix:

$$\left[\begin{array}{ccc|ccc}
\mathrm{Re} z_1^1 & \dots & \mathrm{Re} z_m^1 & -\mathrm{Im} z_1^1 & \dots & -\mathrm{Im} z_m^1 \\
\mathrm{Re} z_1^2 & \dots & \mathrm{Re} z_m^2 & -\mathrm{Im} z_1^2 & \dots & -\mathrm{Im} z_m^2 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
\mathrm{Re} z_1^{k-1} & \dots & \mathrm{Re} z_m^{k-1} & -\mathrm{Im} z_1^{k-1} & \dots & -\mathrm{Im} z_m^{k-1} \\
\mathrm{Re} z_1^k & \dots & \mathrm{Re} z_m^k & -\mathrm{Im} z_1^k & \dots & -\mathrm{Im} z_m^k \\
\hline
\mathrm{Im} z_1^1 & \dots & \mathrm{Im} z_m^1 & \mathrm{Re} z_1^1 & \dots & \mathrm{Re} z_m^1 \\
-\mathrm{Im} z_1^2 & \dots & -\mathrm{Im} z_m^2 & -\mathrm{Re} z_1^2 & \dots & -\mathrm{Re} z_m^2 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
\mathrm{Im} z_1^{k-1} & \dots & \mathrm{Im} z_m^{k-1} & \mathrm{Re} z_1^{k-1} & \dots & \mathrm{Re} z_m^{k-1} \\
-\mathrm{Im} z_1^k & \dots & -\mathrm{Im} z_m^k & -\mathrm{Re} z_1^k & \dots & -\mathrm{Re} z_m^k \\
\hline
\dots & \dots & \dots & \dots & \dots & \dots
\end{array}\right].$$

Now let us write down the $(k \times k)$-matrix $A(k)$. Recall that the j-th row of it is obtained by adding to $(2j-1)$-th row of the above matrix $i = \sqrt{-1}$ times the $(2j)$-th row of it. Then we obtain that $A(k)$ is equal to

$$\left[\begin{array}{ccc|ccc}
\mathrm{Re} z_1^1 + i\mathrm{Re} z_1^2 & \dots & \mathrm{Re} z_m^1 + i\mathrm{Re} z_m^2 & -\mathrm{Im} z_1^1 - i\mathrm{Im} z_1^2 & \dots & -\mathrm{Im} z_m^1 - i\mathrm{Im} z_m^2 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
\mathrm{Re} z_1^{k-1} + i\mathrm{Re} z_1^k & \dots & \mathrm{Re} z_m^{k-1} + i\mathrm{Re} z_m^k & -\mathrm{Im} z_1^{k-1} - i\mathrm{Im} z_1^k & \dots & -\mathrm{Im} z_m^{k-1} - i\mathrm{Im} z_m^k \\
\hline
\mathrm{Im} z_1^1 - i\mathrm{Im} z_1^2 & \dots & \mathrm{Im} z_m^1 - i\mathrm{Im} z_m^2 & \mathrm{Re} z_1^1 - i\mathrm{Re} z_1^2 & \dots & \mathrm{Re} z_m^1 - i\mathrm{Re} z_m^2 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
\mathrm{Im} z_1^{k-1} - i\mathrm{Im} z_1^k & \dots & \mathrm{Im} z_m^{k-1} - i\mathrm{Im} z_m^k & \mathrm{Re} z_1^{k-1} - i\mathrm{Re} z_1^k & \dots & \mathrm{Re} z_m^{k-1} - i\mathrm{Re} z_m^k
\end{array}\right].$$

Let us denote by A the $(m \times m)$-sub-matrix of the above matrix which stays in the upper left part of it, and by B the $m \times m$-sub-matrix which stays in the lower left part of it. Then it is easy to see that

$$A(k) = \begin{bmatrix} A & -\overline{B} \\ B & \overline{A} \end{bmatrix}.$$

Then the function F is equal

$$\det[A(k) \cdot A(k)^t] = \det[A(k)]^2.$$

Let us show that $\det[A(k)] \in \mathbb{R}$. Indeed

$$\overline{\det[A(k)]} = \det \begin{bmatrix} \overline{A} & -B \\ \overline{B} & A \end{bmatrix} =$$

$$\det \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A & -\overline{B} \\ B & \overline{A} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) = \det[A(k)].$$

It remains to show that $F \not\equiv 0$. Take $E_0 := \operatorname{span}_{\mathbb{C}}\{e_1, e_3, e_5, \dots, e_{k-1}\}$. Then $F(E_0) = 1$. q.e.d.

Now we will present another basis in the space of unitarily invariant valuations. As it was mentioned above this basis is more convenient to obtain integral geometric formulas for non-convex sets (see Section 3). Let ${}^{\mathbf{R}}\mathcal{A}\mathrm{Gr}_{k,2n}$ denote the Grassmannian of *affine* real k-dimensional subspaces in $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Let ${}^{\mathbf{C}}\mathcal{A}\mathrm{Gr}_{k,n}$ denote the Grassmannian of *affine* complex k-dimensional subspaces in $\mathbb{C}^n$. Note that ${}^{\mathbf{R}}\mathcal{A}\mathrm{Gr}_{k,2n}$ and ${}^{\mathbf{C}}\mathcal{A}\mathrm{Gr}_{k,n}$ have natural Haar measures which are unique up to a constant. For every nonnegative integers p and k such that $2p \leq k \leq 2n$ let us introduce the following valuations:

$$U_{k,p}(K) = \int_{E \in {}^{\mathbf{C}}\mathcal{A}\mathrm{Gr}_{n-p,n}} V_{k-2p}(K \cap E) \cdot dE.$$

Clearly $U_{k,p} \in \mathrm{Val}_k^{\mathrm{U}(n)}(\mathbb{C}^n)$.

Proposition 2.1.7. *For any nonnegative integers k, p satisfying $2p \leq k \leq n$ one has*

$$U_{k,p} = c_{n,k,p} \cdot \mathbb{D}(C_{2n-k,n-p}),$$

where $c_{n,k,p}$ is a nonzero normalizing constant depending on n, k and p only. Hence the valuations $U_{k,p}$ with $0 \leq p \leq \frac{\min\{k,2n-k\}}{2}$ form a basis of the space $\mathrm{Val}_k^{\mathrm{U}(n)}(\mathbb{C}^n)$.

Proof. Clearly the second statement immediately follows from the first one and Theorem 2.1.1. First we can rewrite the definition of $U_{k,p}$ as follows:

$$U_{k,p}(K) = \int_{F \in {}^{\mathbf{C}}\mathrm{Gr}_{p,n}} dF \cdot \int_{x \in F} dx \cdot V_{k-2p}(K \cap (x + F^{\perp})),$$

where $F^{\perp}$ denotes the orthogonal complement of F. Let us compute the image of $U_{k,p}$ in the space $C({}^{\mathbf{R}}\mathrm{Gr}_{k,n})$ under the imbedding described in Section 1. Fix any $L \in {}^{\mathbf{R}}\mathrm{Gr}_{k,n}$. Let D_L denote the unit Euclidean ball inside L. Then by a straightforward elementary computation one can easily see that for $K = D_L$ the inner integral in the last formula is equal to $c \cdot |\cos(L,F)|$, where c is a normalizing constant. Hence

$$U_{k,p}(D_L) = c \cdot \int_{F\in {}^{\mathbf{C}}\mathrm{Gr}_{p,n}} dF \cdot |\cos(L,F)| \tag{1}$$
$$= c \cdot \int_{F\in {}^{\mathbf{C}}\mathrm{Gr}_{p,n}} dF \cdot |\cos(L^\perp,F^\perp)|$$
$$= c \cdot \int_{E\in {}^{\mathbf{C}}\mathrm{Gr}_{n-p,n}} dE \cdot |\cos(L^\perp,E)|.$$

It is easy to see that for any $M \in {}^{\mathbf{R}}\mathrm{Gr}_{k,2n}$, and for $2k \le l$

$$C_{k,l}(D_M) = c' \cdot \int_{E\in {}^{\mathbf{C}}\mathrm{Gr}_{l,n}} dE \cdot |\cos(M,E)|, \tag{2}$$

where c' is a normalizing constant. Clearly (1) and (2) imply the theorem. q.e.d.

3. Integral geometry in $\mathbb{C}^n$

Using the classification of unitarily invariant valuations obtained in the previous section, we will establish new integral geometric formulas for real submanifolds in $\mathbb{C}^n$. Note that these formulas will be valid not only for convex domains, but for arbitrary piecewise smooth submanifolds of $\mathbb{C}^n$ with corners.

The method to obtain the result for non-convex sets using the convex case is as follows. First one should guess the correct formula for the general case. Next one can approximate nicely piecewise smooth set by polyhedral sets. The last set can be presented as a finite union of convex polytopes. For each convex polytope and for each finite intersection of them we can apply the formulas for the convex case. The final result follows by the inclusion-exclusion principle. In Subsection 3.1 we obtain new integral geometric formulas in $\mathbb{C}^n$. In Subsection 3.2 we compute explicitly the constants in one of these formulas in the particular case $n = 2$. In Subsection 3.3 we discuss another example of unitarily invariant valuation, Kazarnovskii's pseudovolume.

3.1 General results

Let us denote by $\mathrm{IU}(n)$ the group of all isometries of $\mathbb{C}^n$ preserving the complex structure. (Clearly this group is isomorphic to the semidirect product $\mathbb{C}^n \rtimes \mathrm{U}(n)$.)

Note also that the intrinsic volumes V_i in a Euclidean space $\mathbb{R}^N$ can be defined not only for convex compact domains but also for compact domains with piecewise smooth boundary (even more generally, for compact piecewise smooth submanifolds with corners). For instance for a domain Ω with smooth boundary they can be defined as follows: $V_i(\Omega) := \frac{1}{N} M_{N-1-i}(\partial\Omega)$, where for any hypersurface Σ

$$M_r(\Sigma) := \binom{N-1}{r}^{-1} \int_\Sigma \{k_{i_1}, \ldots, k_{i_r}\} d\sigma,$$

where $\{k_{i_1}, \ldots, k_{i_r}\}$ denotes the r-th elementary symmetric polynomial in the principal curvatures $k_{i_1}, \ldots, k_{i_r}$, and $d\sigma$ is the measure induced by the Riemannian metric.

Then we can define the expressions $U_{k,p}(\Omega)$ for $0 \leq 2p \leq k \leq 2n$ (for convex compact sets they were defined in Section 2). The correct generalization is as follows:

$$U_{k,p}(\Omega) = \int_{E \in {}^{\mathbf{C}}\mathcal{A}\mathrm{Gr}_{n-p,n}} V_{k-2p}(\Omega \cap E) \cdot dE,$$

where we use the above definition of $V_{k-2p}(\Omega)$.

Remark. In fact the expressions $U_{k,p}$ can be defined also for compact piecewise smooth submanifolds of $\mathbb{C}^n$ with corners.

Theorem 3.1.1. *Let Ω_1, Ω_2 be compact domains in $\mathbb{C}^n$ with piecewise smooth boundaries such that for every $U \in \mathrm{IU}(n)$ the intersection $\Omega_1 \cap U(\Omega_2)$ has finitely many components. Then*

$$\int_{U \in \mathrm{IU}(n)} \chi(\Omega_1 \cap U(\Omega_2)) dU$$
$$= \sum_{k_1+k_2=2n} \sum_{p_1,p_2} \kappa(k_1, k_2, p_1, p_2) U_{k_1,p_1}(\Omega_1) U_{k_2,p_2}(\Omega_2),$$

where the inner sum runs over $0 \leq p_i \leq \frac{\min\{k_i, 2n-k_i\}}{2}$, $i = 1, 2$, and $\kappa(k_1, k_2, p_1, p_2)$ are certain constants depending on n, k_1, k_2, p_1, p_2 only.

Theorem 3.1.2. *Let Ω be a compact domain in $\mathbb{C}^n$ with piecewise smooth boundary. Let $0 < q < n$, $0 < 2p < k < 2q$. Then*

$$\int_{E \in {}^{C}\mathcal{A}\mathrm{Gr}_{q,n}} U_{k,p}(\Omega \cap E) = \sum_{p=0}^{[k/2]+n-q} \gamma_p \cdot U_{k+2(n-q),p}(\Omega),$$

where the constants γ_p depend only on n, q, and p.

Let us denote by $\mathcal{A}\mathrm{LGr}_n$ the (noncompact) Grassmannian of *affine* Lagrangian subspaces of $\mathbb{C}^n$. Clearly it is a homogeneous space of the group $\mathrm{IU}(n)$.

Theorem 3.1.3. *Let Ω be a compact domain in $\mathbb{C}^n$ with piecewise smooth boundary. Then*

$$\int_{\mathcal{A}\mathrm{LGr}(\mathbb{C}^n)} \chi(E \cap \Omega) dE = \sum_{p=0}^{[n/2]} \beta_p \cdot U_{n,p}(\Omega),$$

where β_p are certain constants depending on n and p only.

Remarks.

1) Theorems 3.1.1 and 3.1.2 are analogs of general kinematic formulas of Poincaré, Chern [11], [13] and Federer [15] (see also [43], especially Ch. 15).

2) These results can be formulated and proved not only for domains but also for piecewise smooth compact submanifolds in $\mathbb{C}^n$ with corners. To do it, consider an ε-neighborhood of this submanifold for small $\varepsilon > 0$. Then apply the above formulas to this domain. Both sides depend polynomially on ε. Comparing the lowest degree terms we get the mentioned generalizations. We do not reproduce here the explicit computations.

3) It would be of interest to compute the constants $\kappa(k_1, k_2, p_1, p_2)$ and β_p in Theorems 3.1.1, 3.1.2, and 3.1.3 explicitly. We could not do it in general. But we compute them in the first nontrivial case $n = 2$ for Theorem 3.1.3 in the next subsection.

3.2 Integral geometry in $\mathbb{C}^2$

In this subsection we will compute explicitly the constants in one of the integral geometric formulas discussed in the previous subsection in the particular case of $\mathbb{C}^2$. In order to do these computations we first recall the classical presentations for the orthogonal group $\mathrm{SO}(4)$ (more precisely for its universal covering $\mathrm{Spin}(4)$) and for the Grassmannian of oriented 2-planes in $\mathbb{R}^4$ which we will denote by ${}^{\mathbf{R}}\mathrm{Gr}^+_{2,4}$.

Let us denote the standard complex structure on $\mathbb{C}^2$ by i. Let us identify $\mathbb{C}^2$ with the quaternionic space $\mathbb{H}$ with the usual anti-commuting complex structures i, j, and $k = ij$. Then clearly $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$.

Recall that the group of quaternions with norm equal to 1 acts by left multiplication on $\mathbb{H} \equiv \mathbb{C}^2$ and thus is identified with the group SU(2).

Also we have the isomorphism

$$\Phi : \mathrm{SU}(2) \times \mathrm{SU}(2)/\{\pm Id\} \tilde{\longrightarrow} \mathrm{SO}(4)$$

defined by

$$\Phi((q_1, q_2))(x) = q_1 x q_2^{-1},$$

where q_1, q_2 are norm one quaternions. Hence we can and will identify the group Spin(4) with $\mathrm{SU}(2) \times \mathrm{SU}(2)$. Let $E_0 \in {}^{\mathbf{R}}\mathrm{Gr}^+_{2,4}$ be the $\mathrm{span}_{\mathbb{R}}\{1, i\}$ with standard orientation coming from the complex structure. Clearly the stabilizer of E_0 in $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ is equal to $T \times T$ where

$$T = \{z \in \mathbb{C} | |z| = 1\} = \mathrm{U}(1) \subset \mathrm{SU}(2).$$

Hence we have the following presentation of the Grassmannian of real oriented 2-planes in $\mathbb{R}^4$:

$${}^{\mathbf{R}}\mathrm{Gr}^+_{2,4} = \mathrm{SU}(2)/T \times \mathrm{SU}(2)/T.$$

However $\mathrm{SU}(2)/T \simeq \mathbb{C}P^1$, where $\mathbb{C}P^1$ is (as usual) the complex projective line. For our computations it will be convenient to identify $\mathbb{C}P^1$ with the 2-dimensional sphere of radius 1/2. Moreover it will be convenient to consider this sphere S^2 in the standard coordinate space $\mathbb{R}^3$ with the center $(1/2, 0, 0)$. Moreover $E_0 \in {}^{\mathbf{R}}\mathrm{Gr}^+_{2,4} = S^2 \times S^2$ will correspond to the point $((1, 0, 0), (1, 0, 0))$. The following lemma can be proved by a straightforward computation.

Lemma 3.2.1. *Let $E = (t_1, t_2) \in S^2 \times S^2 = {}^{\mathbf{R}}\mathrm{Gr}^+_{2,4}$. Let $t_i = (x_i, y_i, z_i)$, $i = 1, 2$. Then $|\cos(E, E_0)| = |x_1 + x_2 - 1|$.*

For $\mathbb{C}^2$ Theorem 2.1.1 says:

Proposition 3.2.2. *For $0 \le k \le 4, k \neq 2$, the space $\mathrm{Val}_k^{\mathrm{U}(2)}(\mathbb{C}^2)$ is spanned by V_k; $\mathrm{Val}_2^{\mathrm{U}(2)}(\mathbb{C}^2)$ is spanned by V_2 and by ϕ, where $\phi(K) = \int_{\xi \in \boldsymbol{C}P^1} \mathrm{vol}_2(\mathrm{Pr}_\xi K) d\xi$.*

Recall that the total measure of $\mathbb{C}P^1$ is chosen to be equal to one. Now let us describe the image of the valuation ϕ in $C({}^{\mathbf{R}}\mathrm{Gr}^+_{2,4})$. Let us denote this image by f.

Lemma 3.2.3. *For every* $E = (t_1, t_2) \in S^2 \times S^2 = {}^{\mathbf{R}}\mathrm{Gr}^+_{2,4}$ *with* $t_i = (x_i, y_i, z_i)$, $i = 1, 2$

$$f(E) = \mathrm{vol}_2 D_2 \cdot \left(\left(x_2 - \frac{1}{2}\right)^2 + \frac{1}{4}\right),$$

where D_2 *denotes the unit* 2*-dimensional Euclidean disk.*

This lemma follows immediately from Lemma 3.2.1 and the fact that the set of complex lines in $\mathbb{C}^2$ is SU(2)-orbit of E_0.

Let us denote by LGr_n the Grassmannian of Lagrangian subspaces in $\mathbb{C}^n$. Let us define a valuation $\psi \in \mathrm{Val}_2^{\mathrm{U}(2)}(\mathbb{C}^2)$ as follows:

$$\psi(K) = \int_{F \in \mathrm{LGr}_2} \mathrm{vol}_2(\mathrm{Pr}_F(K)) dF,$$

where dF is the Haar measure on LGr_2 normalized by 1. The main result of this subsection is:

Theorem 3.2.4.

$$\phi + 2\psi = \frac{\pi}{V_2(D_2)} V_2.$$

Proof. Let

$$\widetilde{\phi} := \frac{1}{\mathrm{vol}_2 D_2} \phi = \frac{1}{\pi}\phi, \ \widetilde{\psi} := \frac{1}{\mathrm{vol}_2 D_2} \psi = \frac{1}{\pi}\psi.$$

Let us denote by $\widetilde{g}$ the image of $\widetilde{\psi}$ in $C({}^{\mathbf{R}}\mathrm{Gr}^+_{2,4})$, and by $\widetilde{f}$ the image of $\widetilde{\phi}$ in $C({}^{\mathbf{R}}\mathrm{Gr}^+_{2,4})$. By Lemma 3.2.3

$$(**) \qquad \widetilde{f}(E) = \left(x_2 - \frac{1}{2}\right)^2 + \frac{1}{4}$$

for every $E = (t_1, t_2) \in S^2 \times S^2 = {}^{\mathbf{R}}\mathrm{Gr}^+_{2,4}$ with $t_i = (x_i, y_i, z_i)$, $i = 1, 2$. Now let us describe $\widetilde{g}$. Thus $\widetilde{f}$ is considered as a function on the second copy of S^2. Let $E_1 = \mathrm{span}\{1, j\} \in \mathrm{LGr}_2$. It is easy to see that $E_1 = U_0(E_0)$, where $U_0 \in \mathrm{SO}(4)$ is defined by

$$U_0(x) = \frac{1+k}{\sqrt{2}} \cdot x \cdot \overline{\frac{1+k}{\sqrt{2}}}$$

for every $x \in \mathbb{H} = \mathbb{R}^4$. Then

$$\begin{aligned}\widetilde{g}(E) &= \int_{F\in \mathrm{LGr}_2} |\cos(F,E)| dF \\ &= \int_{U\in \mathrm{U}(2)} |\cos(U(E_1),E)| dU \\ &= \int_{U\in \mathrm{U}(2)} |\cos(UU_0(E_0),E)| dU \\ &= \int_{U\in \mathrm{U}(2)} |\cos(E_0, U_0^{-1}U(E))| dU.\end{aligned}$$

However $\mathrm{U}(2) = (\mathrm{SU}(2)\times \mathrm{U}(1))/\{\pm 1\}$, where $(q,\lambda) \in \mathrm{SU}(2)\times \mathrm{U}(1)$ acts on $x \in \mathbb{H}$ by $x \mapsto q\cdot x \cdot \lambda^{-1}$. In the formulas below we will write the action of $\lambda \in \mathrm{U}(1)$ on $E \in {}^{\mathbf{R}}\mathrm{Gr}^+_{2,4}$ from the right: $\lambda(E) = E\cdot\lambda^{-1}$. In this notation the last integral can be rewritten as

$$\begin{aligned}&\int_{V\in \mathrm{SU}(2)} dV \int_{\lambda\in \mathrm{U}(1)} d\lambda \cdot \left|\cos\left(E_0, \frac{1-k}{\sqrt{2}} V\cdot E\cdot \lambda^{-1}\cdot \frac{1+k}{\sqrt{2}}\right)\right| \\ &= \int_{V\in \mathrm{SU}(2)} dV \int_{\lambda\in \mathrm{U}(1)} d\lambda \cdot \left|\cos\left(E_0, V\cdot E\cdot \lambda^{-1} \frac{1+k}{\sqrt{2}}\right)\right| \\ &= \int_{\lambda\in \mathrm{U}(1)} d\lambda \cdot \widetilde{f}\left(E\cdot\lambda^{-1}\cdot\frac{1+k}{\sqrt{2}}\right).\end{aligned}$$

By (**) we can write $\widetilde{f}$ as

$$\widetilde{f}(E) = h(E) + \frac{1}{3},$$

where $h(E) = (x_2 - \frac{1}{2})^2 - \frac{1}{12}$. The function h on the sphere S^2 (of radius 1/2) has the property

$$\int_{S^2} h = 0.$$

We have

$$\widetilde{g}(E) = \int_{\lambda\in \mathrm{U}(1)} d\lambda \cdot h\left(E\cdot\lambda^{-1}\cdot\frac{1+k}{\sqrt{2}}\right) + \frac{1}{3}.$$

It is easy to see that

$$h_1(E) := h\left(E\cdot\frac{1+k}{\sqrt{2}}\right) = y_2^2 - \frac{1}{12},$$

$-1/2 \leq y_2 \leq 1/2$. Clearly h_1 is a polynomial of second degree on sphere S^2 such that $\int_{S^2} h_1 = 0$. Hence $\int_{\lambda \in \mathrm{U}(1)} h_1(E \cdot \lambda^{-1})\, d\lambda$ also satisfies these properties, and moreover it is U(1)-invariant. But such a polynomial is unique up to proportionality, hence $\int_{\lambda \in \mathrm{U}(1)} h_1(E \cdot \lambda^{-1})\, d\lambda = c \cdot h(E)$, where c is a constant. Let us compute it. If subspace E is such that $x_2 = 1/2$ then $h(E) = -1/12$. But

$$\int_{\lambda \in \mathrm{U}(1)} h_1(E \cdot \lambda^{-1})\, d\lambda = \frac{1}{2\pi} \int_0^{2\pi} d\phi \left(\frac{\cos^2 \phi}{4} - \frac{1}{12} \right) = 1/24.$$

Hence $c = -1/2$. Thus $\widetilde{g}(E) = -\frac{h(E)}{2} + \frac{1}{3} = -\frac{1}{2}(\widetilde{f}(E) - \frac{1}{3}) + \frac{1}{3} = \frac{1}{2} - \frac{\widetilde{f}(E)}{2}$. Hence $\widetilde{\phi} + 2\widetilde{\psi} = \kappa \cdot V_2$, where κ is a normalizing constant such that for the unit 2-dimensional Euclidean disk D_2, $\kappa \cdot V_2(D_2) = 1$. Thus we get that $\phi + 2\psi = \frac{\pi}{V_2(D_2)} V_2$. q.e.d.

3.3 Kazarnovskii's pseudovolume

In this subsection we discuss another example of unitarily invariant translation invariant continuous valuation which has rather different origin, namely it comes from complex analysis. We discuss so called Kazarnovskii's pseudovolume. The main result of this subsection is a new formula for Kazarnovskii's pseudovolume in integral geometric terms. The proof of this result is based on the classification of unitarily invariant valuations (Theorem 2.1.1).

Now let us recall the definition of Kazarnovskii's pseudovolume following [30], [31]. Let $\mathbb{C}^n$ be Hermitian space with the Hermitian scalar product $(\cdot,\cdot)$. For a convex compact set $K \in \mathcal{K}(\mathbb{C}^n)$ let us denote its supporting functional

$$h_K(x) := \sup_{y \in K} (x, y).$$

For a set $K \in \mathcal{K}(\mathbb{C}^n)$ such that its supporting functional h_K is smooth on $\mathbb{C}^n - \{0\}$ Kazarnovskii's pseudovolume P is defined as follows:

$$P(K) := \int_D (dd^c h_K)^n,$$

where D denotes the unit Euclidean ball on $\mathbb{C}^n$, and $d^c = I^{-1} \circ d \circ I$ for our complex structure I.

Proposition 3.3.1. *Kazarnovskii's pseudovolume P extends by continuity in the Hausdorff metric to all $\mathcal{K}(\mathbb{C}^n)$. Then P is unitarily invariant translation invariant continuous valuation homogeneous of degree n.*

Proof. The first part of the proposition (the continuity) is a standard fact from the theory of plurisubharmonic functions originally due to Chern-Levine-Nirenberg [14] (see also [30], [31]). The unitary invariance, translation invariance, and the homogeneity of degree n are obvious. The only thing which remains to prove is that P is a valuation.

Let A be a convex polytope. It was shown by Kazarnovskii [30] that

$$P(A) = \kappa \sum_F f(F)\gamma(F)\mathrm{vol}_n F,$$

where κ is a normalizing constant, the sum runs over all n-dimensional faces F of A, $\gamma(F)$ is the measure of the exterior angle of A at F, $\mathrm{vol}_n F$ denotes the (n-dimensional) volume of the face F, and $f(F)$ is defined as follows. Let D_F denote the unit ball in the *linear* subspace parallel to the face F. Then $f(F) = \mathrm{vol}(D_F + I \cdot D_F)$. It is easy to see from the above formula that P restricted to the class of convex compact polytopes is a valuation, namely if A_1, A_2, $A_1 \cup A_2$ are convex compact polytopes then

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2).$$

Then it is easy to see that the continuity of P and the valuation property on the subclass of polytopes imply that P is a *weak* valuation on $\mathcal{K}(\mathbb{C}^n)$ (this means that for any real hyperplane H and any $K \in \mathcal{K}(\mathbb{C}^n)$ one has $P(K) = P(K \cap H^+) + P(K \cap H^-) - P(K \cap H)$ where H^+ and H^- denote the half-spaces). However it was shown by Groemer [23] that every continuous weak valuation is valuation (in the usual sense). q.e.d.

The main result of this subsection is as follows.

Theorem 3.3.2.

$$P = \sum_{n/2 \le l \le n} \alpha_l C_{n,l},$$

where $\alpha_l \in \mathbb{R}$ are certain constants depending only on n, and $C_{n,l}$ are valuations defined in the previous section.

Remark. It would be interesting to compute explicitly the constant α_l.

Proof. The proof follows immediately from Proposition 3.3.1 and Theorem 2.1.1. q.e.d.

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